Solving Polynomials, Resolvent Degree, and Geometry

asuther1@uci.edu
alexandersutherland.com
@Alex_Does_Math

Alexander J. Sutherland
University of California, Irvine

HISTORY

A classical problem in mathematics is the following:

Given a polynomial, describe its roots algebraically in terms of its coefficients (as simply as possible).

A more precise version of this question is:

What is the minimal d for which there exists a formula for the generic degree n polynomial in algebraic functions of at most d variables?

Answer is the resolvent degree of the generic degree n polynomial

Denote it by **RD(n)**

FACT

The quadratic, cubic, and quartic formulas imply that RD(n)=1 for $n \leq 4$

Solutions of the quintic (c.f. Bring, Klein) also imply that RD(5)=1

Determining RD(n) is very hard; in fact, there are currently no non-trivial lower bounds on RD(n)

We do, however, have upper bounds on RD(n)

CLASSICAL BOUNDS

Classical work of Hilbert and Sylvester yields

- RD(6) $\leq$ 2,
- RD(7) $\leq$ 3,
- RD(8) $\leq$ 4,
- RD(9) $\leq$ 4,

and Hilbert's Conjectures predict that these inequalities are sharp, i.e.

- RD(6) = 2,
- RD(7) = 3,
- RD(8) = 4,
- RD(9) = 4.

Note that these bounds are really of the form

- RD(n) $\leq$ n-4 for $n \geq 6$, and
- RD(n) $\leq$ n-5 for $n \geq 9$.

THIS PROJECT

Main goals for the project:

- Establish RD(n) $\leq$ n-6 for $n \geq 21$ by fixing the gaps in a paper of G.N. Chebotarev
- Extend these, and other, classical methods to obtain new bounds on RD(n)
- Implement modern results to obtain new bounds on RD(n)

Every reduction of RD(n) would be serious progress.

- Dixmier, 1993

MAIN RESULTS

Our first result is the bound claimed by Chebotarev:

Theorem (Sutherland; 2021)

For n $\geq$ 21, we have that RD(n) $\leq$ n-6.

We prove the n-6 bound using iterated polar cones. Additionally, we use iterated polar cones to establish the following bounds:

Theorem (Sutherland; 2021)

- For n $\geq$ 21, we have that RD(n) $\leq$ n-7.
- For n $\geq$ 109, we have that RD(n) $\leq$ n-8.
- For n $\geq$ 325, we have that RD(n) $\leq$ n-9.
- For 9 $\leq$ m $\leq$ 12 and n > (m-1)!/24, we have that RD(n) $\leq$ n-m.

In joint work with Curtis Heberle, we recover an algorithm of Sylvester, which we use in conjunction with iterated polar cones to prove the following bounds:

Theorem (Heberle, Sutherland; 2021)

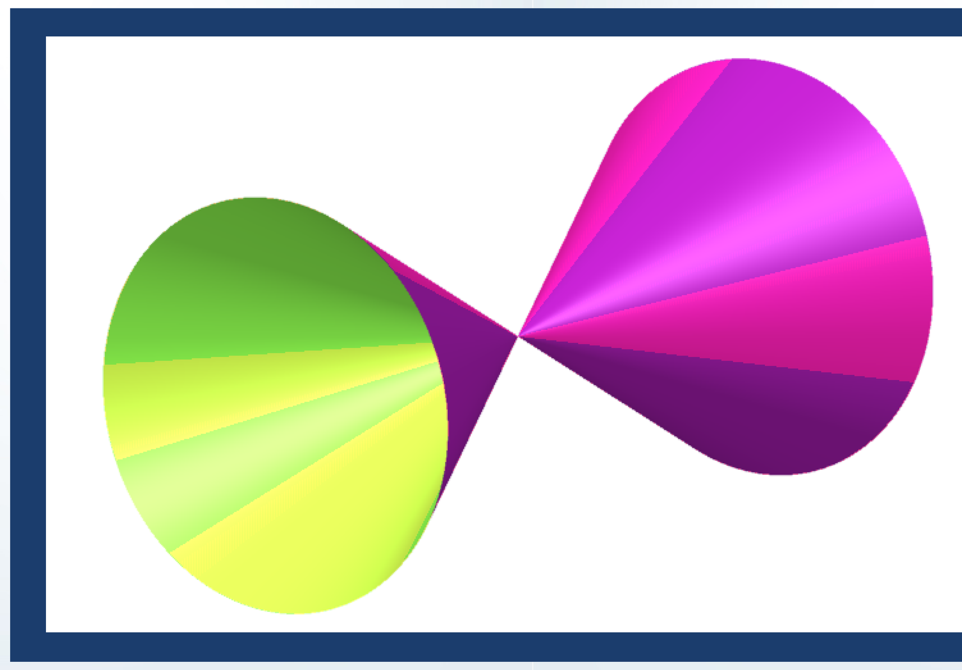
- For n $\geq$ 5,250,198, we have that RD(n) $\leq$ n-13.
- For 14 $\leq$ m $\leq$ 17 and n > (m-1)!/120, we have that RD(n) $\leq$ n-m.
- For n $\geq$ 381,918,437,071,508,900, we have that RD(n) $\leq$ n-22.
- For 23 $\leq$ m $\leq$ 25 and n > (m-1)!/720, we have that RD(n) $\leq$ n-m.

The previous best bounding function $F(m)$ was constructed by Wolfson. We improve upon the method of Wolfson in the general case, as highlighted in the result below.

Theorem (Sutherland; 2021)

We construct a function $G'(m)$ incorporating the above bounds such that

- For all m and $n \geq G'(m)$, we have that RD(n) $\leq$ n-m.
- For each d > 4, $G'(2d^2+4d+4) \leq (2d^2+4d+3)!/d!$. In particular, for n > $(2d^2+4d+3)!/d!$, we have that RD(n) $\leq$ $n-2d^2-4d-4$.
- Denote the previous best bounding function by $F(m)$. Then, $\lim_{m \to \infty} \frac{F(m)}{G'(m)} = \infty$.



POLAR CONES

Start with a degree d hypersurface $H=V(f)$ & a point P

For each $j \in [0,d\text{-}1]$, the jth polar of H at P s a degree d-j hypersurface, denoted by $T(j,f,P)$

The polar cone of H at P, denoted C(H;P), is the intersection of all of the polars of H at P

Key Property

There is a bijection:

Points Q in C(H;P) \ {P}

$\updownarrow$

Lines $\overline{PQ}$ contained in H

Extend in two directions:

- Polar cones for intersections of hypersurfaces, key property above still holds
- Introduce iterated polar cones for analogous property about k-planes

Example of a Second Polar Cone

Second polar cone is

$$C^2(V;P,Q) := C(C(V,P),Q)$$

There is a bijection:

Points R in C(C(V,P),Q) \ $\overline{PQ}$

$\updownarrow$

Planes $\overline{PQR}$ contained in V

THE OBLITERATION ALGORITHM

A Motivating Example

Consider two hypersurfaces H, H' of degrees $d \leq d'$

One way to determine a point of $H \cap H'$ is to solve a polynomial of degree $d \cdot d'$

However, if we can determine a line $\lambda \subseteq H$, then we can determine a point of $\lambda \cap H'$ by solving a degree d' polynomial

We can determine a point P of H by solving a polynomial of degree d. Then, to determine a line $\lambda \subseteq H$, we only need to determine a point Q of C(H;P)\ {P}

Taking polar cones only introduces new hypersurfaces of strictly smaller degree, so we can repeat this process recursively (as long as the ambient projective space is large enough) and will never need to solve polynomials of degree more than d'

CUBIC HYPERSURFACES

A Comparative Example

Consider a smooth cubic hypersurface H in r-dimensional projective over $\mathbb{C}$

- When r=3, H is a smooth cubic surface and contains exactly 27 lines. The resolvent degree of determining such a line is at most 3, as in the work of Farb and Wolfson
- When r=4, H is a smooth cubic 3-fold and every point lies on at least one line. Determining a line on H then has resolvent degree at most RD(6) $\leq$ 2, by determining a point of C(H;P) directly
- When r=5, H is a smooth cubic 4-fold and every point lies on at least one line. We can then solvably determine a line on H by using the obliteration algorithm

SELECTED REFERENCES

Go to tinyurl.com/mw6d79u2 for more information

Chebotarev, *On the Problem of Resolvents*

Dixmier, *Histoire de 13e Problème de Hilbert*

Farb and Wolfson, *Resolvent degree, Hilbert's 13th problem, and geometry*

Heberle and Sutherland, *Upper Bounds on Resolvent Degree via Sylvester's Obliteration Algorithm*

Sutherland, *Upper Bounds on Resolvent Degree and Its Growth Rate*

Sylvester, *On the so-called Tschirnhausen Transformation*

Wiman, *Über die Anwendung der Tschirnhausen-Transformation auf die Reduktion algebraischer Gleichungen.*

Wolfson, *Tschirnhaus Transformations after Hilbert*