Understanding Weight Filtrations via Derived Motivic Measures

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INVARIANTS OF SINGULAR ALGEBRAIC VARIETIES

- **Hironaka's Resolution of Singularities Theorem**: Let k be a field of characteristic 0, and X be a k variety. There exists a smooth projective variety $\tilde{X}$ and a proper birational map $\tilde{X} \to X$ [4].
- **Deligne's Weight Filtration:** Let X be a complex variety. There exist "virtual" Betti numbers $a_i X \in \mathbb{Z}$ for $i \geq 0$ ([5]) such that:
  1. If X is smooth and compact, $a_i X = dim(H^i(X, \mathbb{Q}))$
  2. If $Y \subset X$ is Zariski closed, then $a_i(X) = a_i(Y) + a_i(X - Y)$
- **Gillet-Soulé Weight Filtration:** This weight filtration is constructed using Hironaka's resolution singularities theorem and algebraic K theory (the Gersten Resolution). Further, the filtration recovers that of Deligne [6].
  – This construction also gives a weight filtration on $H^i(X, \mathbb{Z})$

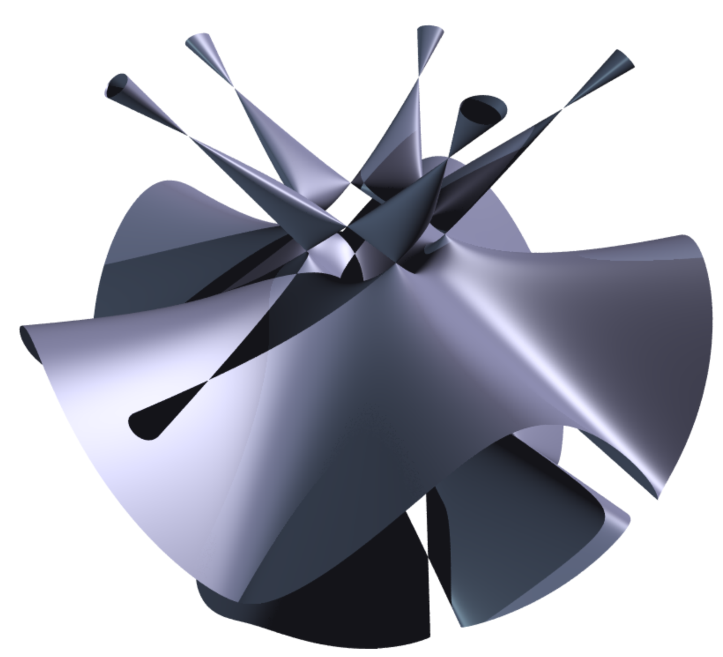

Fig. 1: A complex variety with singularities (source: Wikipedia)

$\mathbf{K_0(\mathcal{V}_k)}$ AND MOTIVIC MEASURES

- Let $\mathcal{V}_k$ denote the category of k varieties, where k is any field
- $K_0(\mathcal{V}_k)$ is defined as the set of isomorphism classes of varieties $[X]$ modulo the equivalence relation $[X] \sim [Y] + [X - Y]$, where $Y \subset X$ is a Zariski closed subvariety.
  – This ring is called the *Grothendieck Ring of Varieties*
- Weight filtrations give rise to **motivic measures**, or ring homomorphisms from $K_0(\mathcal{V}_k)$
- Another example of motivic measures is the point count measure over $\mathbb{F}_q$:

$$\begin{aligned} \mathcal{V}_{\mathbb{F}_q} &\to \textbf{FinSet} \\ X &\mapsto X(\mathbb{F}_q) \end{aligned}$$

THE HIGHER K THEORY OF VARIETIES

- For any field k, Zakharevich constructed a commutative ring spectrum (in the sense of stable homotopy theory) associated to $\mathcal{V}_k$ in [8], denoted $K(\mathcal{V}_k)$, such that $\pi_0(K(\mathcal{V}_k)) = K_0(\mathcal{V}_k)$
- $K(\mathcal{V}_k)$ contains extremely sensitive information that is not recorded in $K_0(\mathcal{V}_k)$
  – $K_1(\mathcal{V}_k)$ is related to birational automorphism classes of varieties [9]
  – If $k = \mathbb{C}$, $K(\mathcal{V}_k)$ is closely related to Waldhausen's algebraic K theory of spaces [2]

LIFTING MOTIVIC MEASURES

- Any reasonable abelian or additive category of motives $\mathcal{M}_k$ admits a motivic measure
- **Updated Statement: Any reasonable abelian or additive category of motives $\mathcal{M}_k$ should admit a map of spectra $K(\mathcal{V}_k) \to K(\mathcal{M}_k)$**
  – Such a map is termed a *derived motivic measure*
- Let $\mathcal{W}$ be a Waldhausen category. Campbell ([2]) gives conditions for when a *motivic measure*:

$$\mu : K_0(\mathcal{V}_k) \to K_0(\mathcal{W})$$

can be lifted to a map of spectra $K(\mu) : K(\mathcal{V}_k) \to K(\mathcal{W})$, where $K(\mathcal{W})$ is the Waldhausen K theory spectrum of $\mathcal{W}$, such that

$$\pi_0(K(\mu)) = \mu : K_0(\mathcal{V}_k) \to K_0(\mathcal{W})$$

MOTIVIC MEASURES OF INTEREST

1. The *mixed Voevodsky measure*

$$\mathcal{V}_k \to DM^{eff}_{gm,t}(k; A)$$

where $DM^{eff}_{gm,t}(k; A)$ is the DG category of geometric effective mixed Voevodsky motives, $t \in \{\text{ét, nis}\}$, and A is any commutative ring.

2. For $char(k) = 0$, the *Gillet-Soulé measure*

$$\mathcal{V}_k \to Ho(Chow^{eff}(k))$$

where $Ho(Chow^{eff}(k))$ is the homotopy category of effective Chow motives over k.

3. The "pointed $\mathbb{A}^1$-unstable homotopy" measure

$$\mathcal{V}_k \to sPre(\mathbb{V}_*; k)_{cdh}$$

where $sPre(\mathbb{V}_*; k)_{cdh}$ is the category of simplicial presheaves of pointed smooth varieties with the cdh topology

RESULTS

Theorem ([1]). *The mixed Voevodsky measure can be lifted for any field k and for any A such that if $char(k) = p$ then $\frac{1}{p} \in A$*

- Composing with realization functors out of $DM^{eff}_{gm,t}(k; A)$ we recover some of the derived measures constructed in [3] along with other ones:

$$\begin{tikzcd} & & K(MHS^A_{eff}) \\ K(\mathcal{V}_k) \arrow[r] & K(DM^{eff}_{gm,t}(k; A)) \arrow[r] \arrow[ur] \arrow[dr] & K(Chow^{eff}(k)) \\ & & K(Mod_{fg}(A)) \end{tikzcd}$$

  – $K(DM^{eff}_{gm,t}(k; A)) \to K(Mod_{fg}(A))$ is the Betti realization,
  – $K(DM^{eff}_{gm,t}(k; A)) \to K(MHS^A_{eff})$ is the mixed Hodge structures realization (for $char(k) = 0$ only)
  – $K(DM^{eff}_{gm,t}(k; A)) \to K(Chow^{eff}(k))$ is a map of K theory spectra built in [7] (for $char(k) = 0$ only)

CURRENT AND FUTURE WORK

Theorem. *The Gillet-Soulé weight complex construction gives a derived motivic measure:*

$$K(\mathcal{V}_k) \to K^{cdh}(s\mathbb{V}^{mb}_*)$$

where $s\mathbb{V}^{mb}_*$ is the subcategory of 'motivically bounded' simplicial pointed smooth projective varieties, equipped with a Waldhausen structure in which cdh maps are weak equivalences.

- We can then build the derived pointed $\mathbb{A}^1$ unstable homotopy measure and construct a new derived Gillet-Soulé measure.
- In the future, we hope to exploit derived motivic measures to uncover more information about classes in $K(\mathcal{V}_k)$, along with guidelines for constructing weight filtrations in different contexts

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