Prequantum Bundle on the Traceless Character Variety of a Surface with Boundary

Mark Ronnenberg [1]

[1] Indiana University - Bloomington

Prequantum Bundles

Let (M, ω) be a symplectic manifold. A *prequantum bundle* over M is a principal $U(1)$-bundle equipped with a connection 1-form whose curvature is a multiple of ω.

Tangles

A *tangle* is a pair (Y, T) where Y is a 3-manifold with boundary and T is a properly embedded 1-dimensional submanifold. If we remove a tubular neighborhood of T, we get a new 3-manifold whose boundary is the union of a punctured surface and a disjoint collection of annuli.

(a) A tangle. (b) Complement of a tangle.

Figure 1. Some tangles.

Traceless Character Varieties

Let F be a compact surface with non-empty boundary. The *traceless character variety* of F, $\mathcal{R}(F)$, is the space of homomorphisms $\pi_1(F) \to SU(2)$ which take loops around boundary components of F to traceless elements of $SU(2)$, modulo conjugation. If F is a 4-punctured sphere, then $\mathcal{R}(F)$ is the *pillowcase*.

The *traceless character variety* of a tangle (Y, T) is the space of homomorphisms $\pi_1(Y \backslash N(T)) \to SU(2)$ which take meridians of T to traceless elements of $SU(2)$, modulo conjugation.

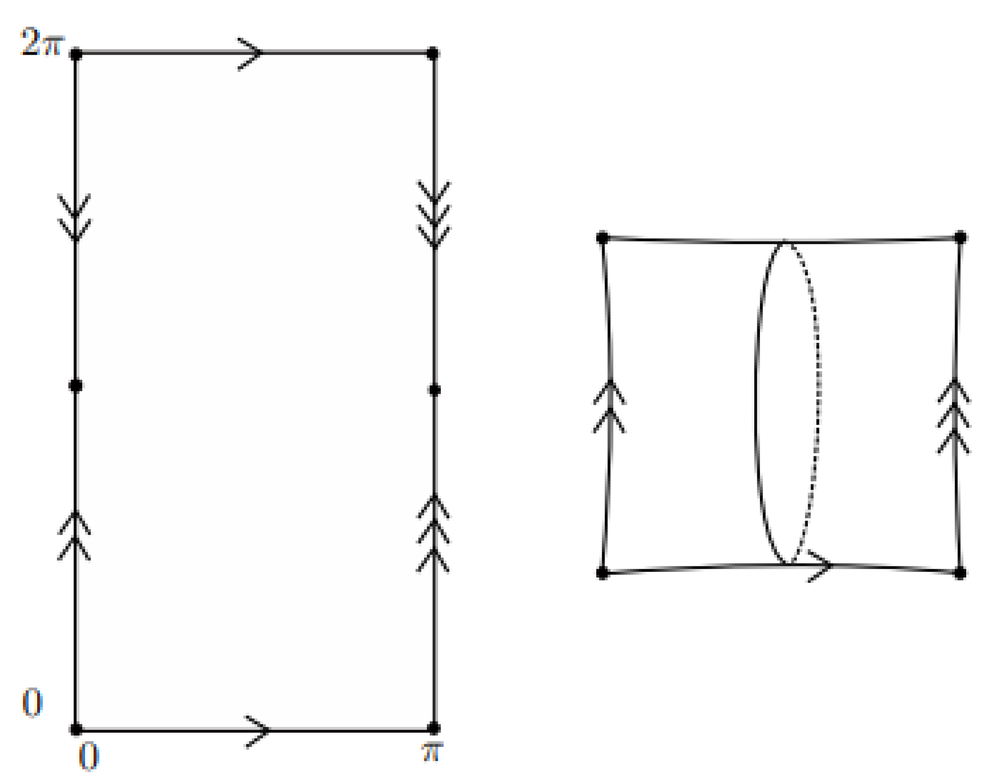

Figure 2. The pillowcase.

Lagrangians from Tangles

$\mathcal{R}(F)$ carries a natural symplectic form. If F is in the boundary of $Y \backslash N(T)$, then $\mathcal{R}(Y, T)$ maps into $\mathcal{R}(F)$ as an immersed Lagrangian.

Kronheimer-Mrowka defined *singular instanton homology*, a gauge-theoretic invariant of links in 3-manifolds. In their construction, traceless character varieties arise as the critical set of the Chern-Simons function.

Hedden-Herald-Kirk initiated the construction of a conjectural Lagrangian intersection theory in the pillowcase, which they dubbed *pillowcase homology*. A version of the *Atiyah-Floer Conjecture* says that singular instanton homology and pillowcase homology are isomorphic.

We want to construct a prequantum bundle over $\mathcal{R}(F)$, which behaves nicely with respect to tangles.

TCV Via Flat Connections

Let

$$\mathcal{A}_{\mathbf{i}}(F) = \left\{ A \in \Omega^1(Y) \otimes \mathfrak{su}(2) \,\middle|\, A|_{\partial F} = \frac{\mathbf{i}}{4} d\theta \right\}$$

$$\mathcal{G}(F) = \left\{ u : F \to SU(2) \,\middle|\, u|_{\partial F_j} = \exp(\mathbf{i} c_j, c_j \in \mathbb{R} \right\}$$

$\mathcal{G}(F)$ acts on $\mathcal{A}_{\mathbf{i}}(F)$ via $u \cdot A = uAu^{-1} - duu^{-1}$.

If we take $\mathcal{A}_{\mathbf{i}}^{\flat}(F)$ to be the subspace of $\mathcal{A}_{\mathbf{i}}(F)$ consisting of *flat* connections, then we have

$$\mathcal{R}(F) \cong \frac{\mathcal{A}_{\mathbf{i}}^{\flat}(F)}{\mathcal{G}(F)}.$$

Prequantum Bundle Over $\mathcal{R}(F)$

Define a map $\Psi : \mathcal{A}_{\mathbf{i}}(F) \times \mathcal{G}(F) \to U(1)$ as follows.

- Let $(A, u) \in \mathcal{A}_{\mathbf{i}}(F) \times \mathcal{G}(F)$.
- Choose a tangle (Y, T) with $\partial(Y \backslash N(T)) = F \cup$ annuli.
- Choose any extensions $\widetilde{A}$ and $\widetilde{u}$ of A and u to $Y \backslash N(T)$.
- Define
$$\Psi(A, u) = \exp\left(2\pi \mathbf{i} \cdot 4k \left(CS_{Y \backslash N(T)}(\widetilde{u} \cdot \widetilde{A}) - CS_{Y \backslash N(T)}(\widetilde{A})\right)\right).$$
- Here, CS_X denotes the *Chern-Simons function*,
$$CS_X(A) = \frac{1}{8\pi^2} \int_X \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$
- Now we define an action of $\mathcal{G}(F)$ on $\mathcal{A}_{\mathbf{i}}(F) \times U(1)$ by $u \cdot (A, z) = (u \cdot A, \Psi(A, u)z)$.
- The orbit space of this action, restricted to $\mathcal{A}_{\mathbf{i}}(F)$, is a principal $U(1)$-bundle over $\mathcal{R}(F)$. In fact, it is a prequantum bundle, i.e. it comes equipped with a connection 1-form whose curvature is a multiple of the natural symplectic form on $\mathcal{R}(F)$.

Ψ is Well-Defined

We have to check that Ψ is well-defined, independent of the choices we made in its definition.

- Make two choices of tangles (Y_i, T_i), with $\partial(Y_i \backslash N(T_i)) = F \cup$ annuli.
- Choose extensions $\widetilde{A}_i$ and $\widetilde{u}_i$ to $Y_i \backslash N(T_i)$.
- Glue $-Y_1$ and Y_2 together along F. This produces a 3-manifold whose boundary is a disjoint union of tori.
- The extended connections and gauge transformations glue together to give a connection and gauge transformation over the glued 3-manfiold with torus boundary.
- The Chern-Simons calculation over the 3-manifold with torus boundary reduces to an integration over the tori. The result is 1/4 the degree of the gauge transformation, viewed as a map $S^1 \to U(1) \cong S^1$.
- So we must multiply the Chern-Simons function by $4k$ to obtain an integer result from this calculation, which guarantees that Ψ is well-defined.

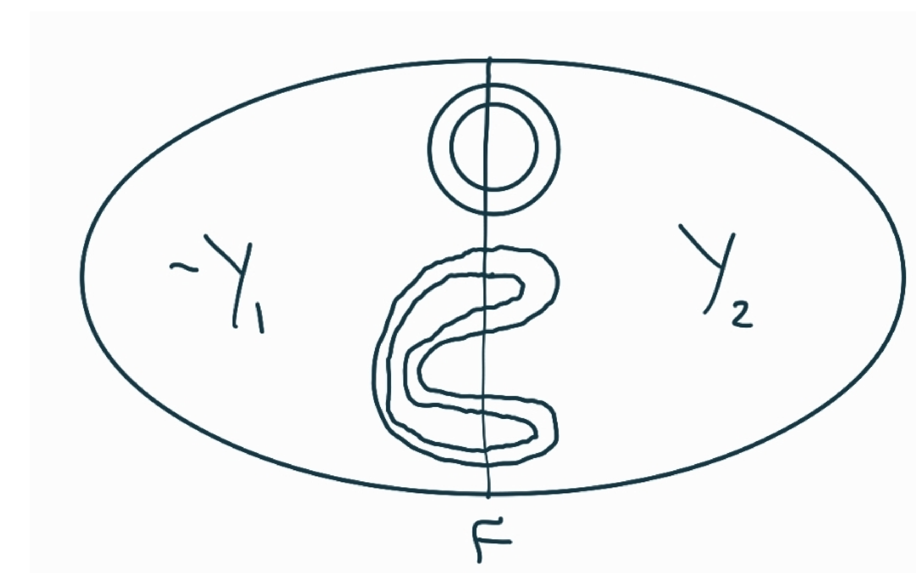

Figure 3. The 3-manifold with torus boundary obtained by gluing $-Y_1$ and Y_2 along F, the common punctured surface in their boundaries.

Legendrian Lifts via CS

The connection 1-form on the prequantum bundle endows its total space with the structure of a contact manifold. Given a tangle (Y, T), with $F \subset \partial(Y \backslash N(T))$, the Chern-Simons function induces a parallel section of the prequantum bundle over the image of $\mathcal{R}(Y, T)$ in $\mathcal{R}(F)$.

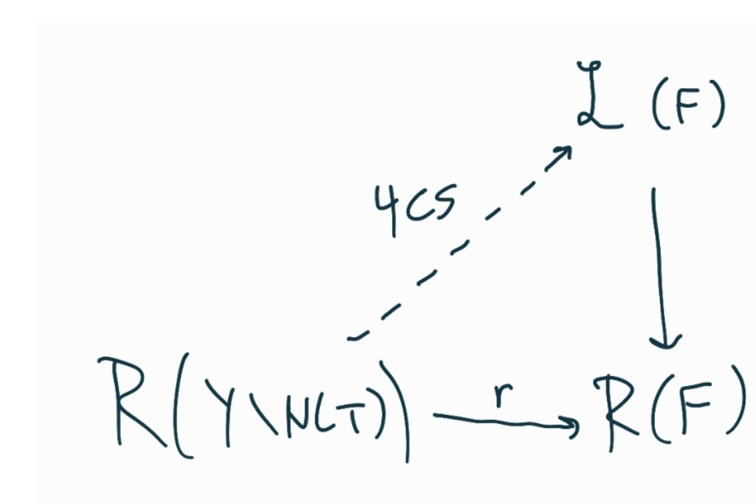

Figure 4. The Chern-Simons function acts as a Legendrian lift of $\mathcal{R}(Y, T)$ to the total space of the prequantum bundle over $\mathcal{R}(F)$.

Future Questions

- Study Legendrian lifts for Lagrangian correspondences induced by holonomy perturbations and earrings.
- Can we obtain invariants from Legendrian lifts (classical invariants, contact homology)?

GSTGC 2022, Georgia Tech

maronnen@iu.edu